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A resummation formula for collapse and revival in the Jaynes–Cummings model

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Abstract

We present a new approach to the study of the function of atomic inversion in the model of interaction of a single two-level atom with a single mode of the quantized electromagnetic field in the coherent state in an ideal resonator. The approach suggested is based on an application of certain number-theoretic results to the approximation of the trigonometric sums of a special form, in particular of the functional equation of the Jacobi theta function. New asymptotic formulae for the atomic inversion are found. The asymptotics that we obtain make it possible to predetermine the details of the behavior of the inversion on various time intervals depending on the parameters of the system.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

1.1. Statement of the problem

The Jaynes–Cummings model (JCM) [6] describes a single two-level atom interacting with a single mode of the quantized electromagnetic field in an ideal resonator [1–5, 15–21]; see also [8, 14]. The great advantage of the JCM is that it is an exactly solvable and at the same time an adequate model. At present, the JCM occupies a special place in quantum optics because it helps to examine and verify the conjectures about more complicated models which are close to real processes.

To study the evolution in time of the atom–field system in the JCM, one often uses such an experimentally observed value as atomic inversion (the difference between the population in the excited state and the ground state of the atom). Let the atom be in its excited state at

[3] Deceased.

$t = 0$ and the radiation field be in a coherent state with the Poissonian photon statistics. Then the atomic inversion $W = W(t)$ at a time $t > 0$ is defined by the following formula ([18]; see also [17]):

$$W(t) = \sum_{n=0}^{+\infty} \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} \left(\frac{\Delta^2}{\Omega_n^2} + \frac{4g^2(n+1)}{\Omega_n^2} \cos(\Omega_n t) \right), \tag{1}$$

where $\Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)}$ is the Rabi frequency, Δ is the detuning parameter, that is, the difference between the frequency of the atomic transition and the frequency of the field in the resonator, $\Delta \geqslant 0$, g is the interaction strength between the atom and the radiation and $g > 0$.

Here, α is in general a complex parameter and $|\alpha|^2$ is the initial average number of photons before the interaction of the field with the atom.

The sum (1) is rather complicated for studying the characteristics of the process of inversion and for its evaluation. For this reason, one takes a function, which approximates (1) well enough, and considers it as the inversion.

A similar problem was considered in [17], where the sum (1) is replaced by one integral (without estimating the remainder term), which is then calculated by the saddle point method, and also in [5] (see also [2]), where the sum of form (1) with $\Delta = 0$ is replaced by the Poisson formula by an infinite sum of integrals, which are then calculated by the saddle point method.

In this paper, to approximate the sum (1), a new method is suggested, which is based on the techniques used in number theory. To get an asymptotic approximation of inversion (1) which is as precise as possible, we investigate the series (1) and approximate it by various functions. In this way, we establish relations between the parameters of the system guaranteeing the appropriateness of the asymptotics obtained. In [8], this problem was solved for the case $\Delta = 0$. We use certain results obtained in [8]. At the same time, when $\Delta \neq 0$, quite new effects are detected in the behavior of the atomic inversion, which are discussed in this paper.

1.2. Introducing new parameters

Since

$$\sum_{n=0}^{+\infty} \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} = 1,$$

then $W(0) = 1$ and $|W(t)| \leqslant W(0) = 1$.

We set

$$A = \sum_{n=0}^{+\infty} \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} \frac{\Delta^2}{\Delta^2 + 4g^2(n+1)},$$

$$W_1(t) = \sum_{n=0}^{+\infty} \frac{|\alpha|^{2n} e^{-|\alpha|^2}}{n!} \frac{4g^2(n+1)}{\Delta^2 + 4g^2(n+1)} \cos(\Omega_n t),$$

so that $W(t) = A + W_1(t)$. We introduce the following new notation:

$$a = \frac{\Delta^2}{4g^2}, \qquad T = 2gt.$$

We emphasize that new parameters a and T are dimensionless. Thus, we will measure time in units of $\frac{1}{2g}$ (note that dimensionless time is used in many works; in addition, in [17] time is measured in units $\frac{\pi}{g}$ and in [1, 18] it is measured in units $\frac{1}{g}$). Sometimes dimensionless time is called [1] the 'scaled time'. By analogy, in what follows, we shall call parameter a the quadratic 'scaled detuning parameter'. With new parameters, we obtain

$$\Omega_n = 2g\sqrt{a+1+n}.$$

$$W_1(t) = U_1(T) = \mathrm{e}^{-|\alpha|^2} \sum_{n=0}^{+\infty} \frac{|\alpha|^{2n}}{n!} \frac{1}{1+\frac{a}{n+1}} \cos\left(T\sqrt{a+1+n}\right), \tag{2}$$

$$A = \sum_{n=0}^{+\infty} \frac{|\alpha|^{2n}}{n!} \mathrm{e}^{-|\alpha|^2} \frac{a}{a+n+1}, \tag{3}$$

$$W(t) = U(T) = U_1(T) + A. \tag{4}$$

The value A does not depend on t and will be calculated below with good accuracy. We note that

$$A = 1 - W_1(0). \tag{5}$$

Let us denote

$$m_1 = |\alpha|^2.$$

In what follows, as in [8], to simplify the calculations for estimating, we shall assume that our main increasing parameter m_1 satisfies the condition

$$m_1 \geqslant 100, \qquad m_1 \text{ is an integer}. \tag{6}$$

1.3. Sketched plan of this paper and certain definitions

We will study the behavior of function $W_1(t)$. With the help of the transformations of the summands of the series (2), we will approximate $U_1(T)$ on the interval of variation of T, which is of the form

$$0 \leqslant T \leqslant (m_1 + a + 1)^{\frac{5}{6}}, \tag{7}$$

by the well-known [11] Jacobi theta series (see also [8]). After that, by applying the functional equation, which is satisfied by this series, we will see that $U_1(T)$ is approximated with good accuracy by a sufficiently simple function. So the properties of this function will determine the behavior of $U_1(T)$ on the indicated interval of variation of T.

We note that the constants in the obtained asymptotic equalities can be replaced by smaller ones if more precise calculations are made. However, we do not present such calculations here because they are very awkward. At the same time, the meaning of these constants is a secondary one. We calculate them mostly to show that the derived asymptotic formulae are effective ones. In reality, the accuracy of the formulae, which we developed, can be much better than what is established in our statements. This can make it possible to extend the limits of the availability of the obtained asymptotics defined by (7). The proximity of certain characteristics of the plots of two approximations: of the principal term of the asymptotics obtained and of the direct sum of a great total number of summands of the series (2), on the time intervals which are essentially longer than (7), indicates the possibility of such an extension for certain values of parameters. We will demonstrate it in subsection 4.2.

Below, we use the following notation:

$\theta, \theta_1, \theta_2, \ldots$—functions whose modulus does not exceed 1; note that in different formulae, they are, generally speaking, different.

For real x, the function $y = \{x\}$ is the *fractional part of the number* x, that is, $y = \{x\} = x - [x]$, where $[x]$ is *the integral part of* x, that is, an integer such that

$[x] \leqslant x < [x] + 1$; the function $y = \|x\| = \min(\{x\}, 1 - \{x\})$ is the *distance from x to the nearest integer*.

We note that, for any x, $0 \leqslant \{x\} < 1$, $0 \leqslant \|x\| \leqslant \frac{1}{2}$.

2. The proof of the main asymptotics for inversion

2.1. Gaussian approximation to Poisson statistics

The formulae presented below are similar to the formulae proved in [8]. Applying the Stirling formula to $n!$, we prove that

$$U_1(T) = \frac{1}{\sqrt{2\pi m_1}}\left(1 - \frac{\theta}{12m_1}\right)F_1(T), \tag{8}$$

where

$$F_1(T) = \sum_{m=0}^{\infty} r(m)\frac{1}{1 + \frac{a}{m+1}}\cos\left(T\sqrt{a+1+m}\right), \tag{9}$$

$$r(m) = \begin{cases} \frac{(m+1)\cdots(m_1-1)}{m_1^{m_1-m-1}}, & \text{if} \quad m < m_1 \\ 1, & \text{if} \quad m = m_1 \\ \frac{m_1^{m-m_1}}{(m_1+1)\cdots m}, & \text{if} \quad m > m_1. \end{cases}$$

The infinite series (9) is approximated with good accuracy by a finite sum. The accuracy of the approximation increases when the total number of summands of the approximating sum grows. We shall denote this total number of summands by $2\nu_1 + 1$ and assume that ν_1 is an arbitrary integer from the interval

$$1 < \nu_1 \leqslant \tfrac{1}{2}m_1$$

(later on, we shall define ν_1 precisely). Then for $F_1(T)$, the following formula is valid:

$$F_1(T) = F_2(T) + 4\theta\frac{m_1}{\nu_1}\left(1 + \frac{a}{m_1}\right)^{-1}\exp\left(-\frac{\nu_1^2}{4m_1}\right), \tag{10}$$

where

$$F_2(T) = \sum_{-\nu_1 \leqslant \nu \leqslant \nu_1} r(m_1+\nu)\left(1 + \frac{a}{m_1+\nu+1}\right)^{-1}\cos\left(T\sqrt{m_1+a+1+\nu}\right). \tag{11}$$

From (10) we see that if ν_1 is any number with the condition

$$\nu_1 \geqslant 2C\sqrt{m_1 \ln m_1}, \qquad C = \text{const} > 0,$$

then the remainder in (10) is less in absolute value than

$$4\frac{m_1}{\nu_1}\left(1 + \frac{a}{m_1}\right)^{-1} m_1^{-C^2},$$

that is, $F_1(T)$ is 'very well' approximated by the sum $F_2(T)$, with the number of summands of order $\sqrt{m_1 \ln m_1}$.

The next step of our transformations is to approximate the coefficients of the function $r(m_1+\nu)$, $|\nu| \leqslant \nu_1$, by the exponential function. It is proved that if

$$1 < \nu_1 \leqslant \sqrt[3]{\tfrac{1}{2}m_1^2}, \tag{12}$$

then we obtain by applying the Euler summation formula to $r(m)$ from (9) (see [8][lemma 6]) that for $|\nu| \leqslant \nu_1$, the following asymptotic formula is valid:

$$r(m_1+\nu) = \exp\left(-\frac{\nu^2}{2m_1}\right)\left(1+\theta_1|\nu|^3 m_1^{-2}+\theta_2|\nu| m_1^{-1}\right). \tag{13}$$

From (11) and (13) with condition (12), we get

$$F_2(T) = \sum_{-\nu_1\leqslant\nu\leqslant\nu_1} \exp\left(-\frac{\nu^2}{2m_1}\right)\left(1+\frac{a}{m_1+\nu+1}\right)^{-1}\cos\left(T\sqrt{m_1+a+1+\nu}\right) + 6\theta_3\left(1+\frac{a}{m_1}\right)^{-1}. \tag{14}$$

The coefficient $\left(1+\frac{a}{m_1+\nu+1}\right)^{-1}$ in (14) for $|\nu| \leqslant \nu_1$ is 'a little' different from the number $\left(1+\frac{a}{m_1+1}\right)^{-1}$, which does not depend on ν. Using this consideration, we get after simple calculations

$$F_2(T) = \left(1+\frac{a}{m_1+1}\right)^{-1}\sum_{-\nu_1\leqslant\nu\leqslant\nu_1} \exp\left(-\frac{\nu^2}{2m_1}\right)\cos\left(T\sqrt{m_1+a+1+\nu}\right) + 10\theta_4\left(1+\frac{a}{m_1}\right)^{-1}. \tag{15}$$

2.2. *Quadratic approximation to the cosine argument*

Now we substitute the argument of the cosine by the quadratic polynomial in ν and estimate the error of such a substitution. Now set

$$m_2 = m_1+a+1, \qquad \beta_0 = T\sqrt{m_2}, \qquad \beta_1 = \frac{T}{2\sqrt{m_2}}, \qquad \beta_2 = \frac{T}{8\sqrt{m_2^3}}.$$

Using the Taylor and the Lagrange formulae, we obtain consecutively

$$\begin{gathered} T\sqrt{m_1+a+1+\nu} = T\sqrt{m_2+\nu} = \beta_0+\beta_1\nu-\beta_2\nu^2+\tfrac{1}{8}\theta_3 T|\nu|^3 m_2^{-\frac{5}{2}}, \\ \cos\left(T\sqrt{m_2+\nu}\right) - \cos\left(\beta_0+\beta_1\nu-\beta_2\nu^2\right) = \tfrac{1}{8}\theta_4 T|\nu|^3 m_2^{-\frac{5}{2}}. \end{gathered} \tag{16}$$

Substituting (16) into (15) and using the estimate

$$\sum_{-\nu_1\leqslant\nu\leqslant\nu_1} |\nu|^3 \exp\left(-\frac{\nu^2}{2m_1}\right) \leqslant 2\int_0^\infty x^3 \mathrm{e}^{-\frac{x^2}{2m_1}}\,\mathrm{d}x = 4m_1^2,$$

we get for $F_2(T)$

$$F_2(T) = \left(1+\frac{a}{m_1+1}\right)^{-1}\left(F_3(T)+\frac{1}{2}\theta_1 T m_1^2 m_2^{-\frac{5}{2}}+10\theta_2\right), \tag{17}$$

where

$$F_3(T) = \sum_{-\nu_1\leqslant\nu\leqslant\nu_1} \exp\left(-\frac{\nu^2}{2m_1}\right)\cos\left(\beta_0+\beta_1\nu-\beta_2\nu^2\right). \tag{18}$$

2.3. Approximation of the finite sum by the Jacobi theta function

From (18), one can see that $F_3(T)$ is a 'principal part' of the theta series $F_4(T)$:

$$F_3(T) = F_4(T) + 10\theta_1 \frac{m_1}{\nu_1} \exp\left(-\frac{\nu_1^2}{2m_1}\right), \tag{19}$$

where

$$F_4(T) = \sum_{\nu=-\infty}^{+\infty} \exp\left(-\frac{\nu^2}{2m_1}\right) \cos\left(\beta_2\nu^2 - \beta_1\nu - \beta_0\right).$$

Note that the greater the T, the worse the remainder term in (17). This is because we approximate the argument of the cosine by a polynomial of the second degree. For the remainder term to be 'good' also for 'great' T, one should approximate the argument of the cosine by a polynomial of degree $n > 2$, but $F_3(T)$, obtained in this way will be not a part of a theta series, and it is not known how to sum it up.

2.4. Apply the resummation formula

Now we apply a functional equation to the theta function $F_4(T)$, which it is also possible to write in the following way:

$$F_4(T) = \operatorname{Re} \sum_{\nu=-\infty}^{+\infty} \exp\left(-\frac{\nu^2}{2m_1}\right) \exp\left(\mathrm{i}\left(\beta_2\nu^2 - \beta_1\nu - \beta_0\right)\right).$$

The functional equation of an arbitrary Θ-function is as follows: if $\operatorname{Re}(\tau) > 0$, and

$$\Theta(\tau, \rho) = \sum_{n=-\infty}^{+\infty} \exp\left(-\pi\tau(n+\rho)^2\right),$$

then

$$\Theta\left(\frac{1}{\tau}, \rho\right) = \sqrt{\tau} \sum_{n=-\infty}^{+\infty} \exp\left(-\pi\tau n^2 + 2\pi\mathrm{i}\rho n\right).$$

Taking the numbers τ and ρ from the equalities

$$-\pi\tau = -\frac{1}{2m_1} + \mathrm{i}\beta_2, \qquad 2\pi\mathrm{i}\rho = -\mathrm{i}\beta_1,$$

after simple calculations we obtain for $F_4(T)$ the formula

$$F_4(T) = T_2^{-\frac{1}{4}} \sum_{n=-\infty}^{+\infty} \exp\left(-\frac{1}{2m_1T_2}\left(n - \frac{\beta_1}{2\pi}\right)^2\right) \cos\left(\frac{\beta_2}{T_2}\left(n - \frac{\beta_1}{2\pi}\right)^2 + \frac{\varphi_1}{2} - \beta_0\right), \tag{20}$$

where

$$\begin{aligned} T_2 &= \frac{1}{4\pi^2 m_1^2} + \frac{\beta_2^2}{\pi^2} = \frac{1}{4\pi^2 m_1^2} + \frac{T^2}{64\pi^2 m_2^3}, \\ \varphi_1 &= \arctan \frac{Tm_1}{4\sqrt{m_2^3}} = \frac{\pi}{2} - \arctan \frac{4\sqrt{m_2^3}}{Tm_1}. \end{aligned} \tag{21}$$

Setting now in (19) $\nu_1 = \left[\sqrt[3]{m_1^2/2}\right]$ and combining formulae (7), (10), (17), (19) and (20), we obtain for $U_1(T)$ the following asymptotic formula:

$$U_1(T) = \frac{1}{\sqrt{2\pi m_1}}\left(1 - \frac{\theta}{12m_1}\right)\left(1 + \frac{a}{m_1+1}\right)^{-1}\left(T_2^{-\frac{1}{4}}\sum_{n=-\infty}^{+\infty}\exp\left(-\frac{1}{2m_1T_2}\left(n - \frac{\beta_1}{2\pi}\right)^2\right)\right.$$
$$\left.\times\cos\left(\frac{\beta_2}{T_2}\left(n - \frac{\beta_1}{2\pi}\right)^2 + \frac{\varphi_1}{2} - \beta_0\right) + \theta_1 T m_1^2 m_2^{-\frac{5}{2}} + 10\theta_2\right). \tag{22}$$

Analyzing (22) we see that the principal term in it is the term which is obtained with the integer $n = n_1$ that is closest to the number $\frac{\beta_1}{2\pi}$.

Now assume that T satisfies condition (7) and besides

$$\gamma = \left\|\frac{\beta_1}{2\pi}\right\| = \left\|\frac{T}{4\pi\sqrt{m_2}}\right\|, \tag{23}$$

$$D = \frac{1}{2m_1T_2} = \frac{1}{2m_1}\left(\frac{1}{4\pi^2m_1^2} + \frac{\beta_2^2}{\pi^2}\right)^{-1}. \tag{24}$$

Then for $U_1(T)$, the following formula is valid:

$$U_1(T) = \frac{1}{\sqrt{2\pi m_1}}\left(1 - \frac{\theta}{12m_1}\right)\left(1 + \frac{a}{m_1+1}\right)^{-1}\left(T_2^{-\frac{1}{4}}\exp(-D\gamma^2)\right.$$
$$\left.\times\cos\left(2m_1\beta_2D\gamma^2 + \frac{\varphi_1}{2} - \beta_0\right) + \theta_1\left(Tm_1^2m_2^{-\frac{5}{2}} + 12\right)\right). \tag{25}$$

Formula (25) is obtained from (22); the principal term with $n = n_1$ is isolated, where

$$n_1 = \left[\frac{\beta_1}{2\pi}\right], \qquad \text{if} \quad \left\{\frac{\beta_1}{2\pi}\right\} \leqslant \frac{1}{2},$$
$$n_1 = \left[\frac{\beta_1}{2\pi}\right] + 1, \qquad \text{if} \quad \left\{\frac{\beta_1}{2\pi}\right\} > \frac{1}{2},$$

and the sum of the remaining summands is estimated trivially as

$$\sum_{\substack{n=-\infty \\ n\neq n_1}}^{+\infty}\exp\left(-\frac{1}{2m_1T_2}\left(n - \frac{\beta_1}{2\pi}\right)^2\right) \leqslant 2\left(\exp\left(-\frac{\pi^2}{4}m_1\right) + \exp\left(-4\pi^2m_2^{\frac{1}{3}}\right)\right).$$

3. Discussion of the asymptotic behavior of the atomic inversion for various time intervals

3.1. General remarks

We study the behavior of function $U_1(T)$ when T varies from 0 to $m_2^{\frac{5}{6}}$.

Traditionally, the behavior of the inversion is described as a sequence of collapses and revivals of quantum oscillations. However, sometimes it is more convenient to consider a packet of oscillations as a whole, with both the revival and the collapse (exclusion is the first packet; it is of the collapsed oscillations only). Every packet has its own characteristics: the maximal by absolute value oscillation, the time position of the maximal by absolute value oscillation (‘the position of the peak of the packet’; it is also called the ‘time of revival’), the time interval where the oscillations of the packet have a notable value, etc.

Formula (25) will be appropriate (with a packet) when its principal term, that is, the summand with the cosine, will take values much greater (by absolute value) than the remainder. By assuming (later it will be proved rigorously) that on the studied interval of varying T, say, for $\Upsilon \leqslant T \leqslant \Upsilon + h$ the cosine takes all values from -1 to $+1$, let us see how the principal term behaves in this case. It is not difficult to prove that for T from (7) and (21) the inequality

$$T_2^{-\frac{1}{4}} \gg T m_1^2 m_2^{-\frac{5}{2}} + 12$$

always holds and, besides, the number D from (24) is always 'great', namely

$$D \geqslant \min\left(\pi^2 m_1, 16\pi^2 m_2^{\frac{1}{3}}\right).$$

For this reason, if in (25) the value of γ is 'great', then $\exp(-D\gamma^2)$ is a 'very small' number and the principal term is smaller than the remainder. More precisely, if

$$D\gamma^2 \geqslant 2\ln m_2,$$

then it is easy to get the inequality

$$T_2^{-\frac{1}{4}} \exp(-D\gamma^2) \leqslant T m_1^2 m_2^{-\frac{5}{2}} + 12.$$

That is why for (25) to be rich in content, it is necessary that γ should be 'small', namely that the inequality

$$\gamma \leqslant \sqrt{\frac{2\ln m_2}{D}} \tag{26}$$

should be valid.

3.2. *Behavior of the asymptotics around revival time*

From the definition of γ (see (23)), we conclude that the asymptotics (25) will be meaningful if the numbers

$$\frac{T}{4\pi\sqrt{m_2}}$$

are 'close' to integers. Since $0 \leqslant T \leqslant m_2^{\frac{5}{6}}$, then one needs to consider such T which are 'close' to the numbers of the form

$$Tr_k = k4\pi\sqrt{m_2}; \qquad k = 0, 1, \ldots, k_1; \qquad k_1 = \left[\frac{m_2^{\frac{1}{3}}}{4\pi}\right]. \tag{27}$$

By analogy with the resonance case, we can call Tr_k, by which the position of the peak of the packet of quantum oscillations is defined, 'the time of the kth revival'.

3.3. *The bounds for the first packet (the first collapse)*

Now let us consider a part of interval (7), namely

$$0 \leqslant T \leqslant 2\pi\sqrt{m_2}. \tag{28}$$

Then

$$\gamma = \frac{T}{4\pi\sqrt{m_2}}.$$

For the nontriviality of (25), one needs the validity of inequality (26) or, in the case considered, of the inequality

$$T \leqslant \sqrt{\frac{32\pi^2 m_2 \ln m_2}{D}}.$$

On the other hand, on interval (28) for D the following estimate is valid:

$$D \geqslant \frac{1}{2m_1}\left(\frac{1}{4\pi^2 m_1^2} + \frac{1}{16m_2^2}\right)^{-1} \geqslant m_1\left(\frac{1}{2\pi^2} + \frac{1}{8}\right)^{-1} \geqslant 4m_1.$$

Thus, for the nontriviality of formula (25) on interval (28), it is necessary that the following inequality holds:

$$T \leqslant \sqrt{8\pi^2 \frac{m_2}{m_1} \ln m_2}. \tag{29}$$

So, if $0 \leqslant T \leqslant 2\pi\sqrt{m_2}$, then only for T situated in the left end of this interval, namely for T which are satisfied by (29), can the asymptotics (25) be rich in content. Note that for large detuning, when $a \gg 1$,

$$\sqrt{8\pi^2 \frac{m_2}{m_1} \ln m_2} \gg 1.$$

At the same time, if

$$2\pi\sqrt{m_2} \geqslant T > \sqrt{8\pi^2 \frac{m_2}{m_1} \ln m_2},$$

then the principal term of (25) is less than the remainder and (25) gives only an upper bound for $U_1(T)$.

It is natural to ask the following question: for which T does the first summand of (25) take values that are certainly greater by the absolute value than the remainder? It is easy to find an answer to this question. Since for D the upper bound of the form $D \leqslant 2\pi^2 m_1$ holds, then for $D\gamma^2 \leqslant 1$, that is, certainly for $\gamma^2 \leqslant \frac{1}{2\pi^2 m_1}$, or for

$$0 \leqslant T \leqslant 2\sqrt{2}\sqrt{\frac{m_2}{m_1}}, \tag{30}$$

the estimate $\exp(-D\gamma^2) \geqslant \exp(-1)$ is valid. If the cosine takes all possible values from -1 to $+1$ on interval (30), then the principal term in (25) will also take values much greater by the absolute value than the remainder, that is, the asymptotics (25) will be useful. Let us look at the argument of the cosine in (25), which is convenient from the cosine evenness to be rewritten as

$$\Phi(T) = T\sqrt{m_2} - \frac{1}{2}\arctan\frac{Tm_1}{4m_2^{\frac{3}{2}}} - \frac{T^3}{128\pi^2 m_2^{\frac{5}{2}}}\left(\frac{1}{4\pi^2 m_1^2} + \frac{T^2}{64\pi^2 m_2^3}\right)^{-1}. \tag{31}$$

On interval (30), the principal summand of (31) is the first summand. The sum of the other two is less by the absolute value than $\frac{3}{2\sqrt{2}}\frac{\sqrt{m_1}}{m_2}$. That is why, for any $h > 0$, we find

$$\Phi(T+h) - \Phi(T) = h\sqrt{m_2} + \theta\frac{3}{\sqrt{2}}\frac{\sqrt{m_1}}{m_2}.$$

Consequently, $\Phi(T+h) - \Phi(T) > \pi$, if

$$h > \frac{\pi}{\sqrt{m_2}} + \frac{3}{\sqrt{2}}\sqrt{\frac{m_1}{m_2}}\frac{1}{m_2} = \varkappa.$$

We obtained that for $0 \leqslant T \leqslant 2\sqrt{2}\sqrt{\frac{m_2}{m_1}}$, at any interval of the form $(T, T+\varkappa)$ the cosine takes values from -1 to $+1$, and

$$\frac{1}{\varkappa}2\sqrt{2}\sqrt{\frac{m_2}{m_1}} \approx \frac{2\sqrt{2}}{\pi}\frac{m_2}{\sqrt{m_1}}$$

such intervals there. Moreover, the factor by the cosine in (25),

$$T_2^{-\frac{1}{4}} \exp(-D\gamma^2) = \exp(-D\gamma^2)\left(\frac{1}{4\pi^2 m_1^2} + \frac{T^2}{64\pi^2 m_2^3}\right)^{-\frac{1}{4}},$$

does not vary practically.

Summing up briefly the conclusions on the behavior of the function $U_1(T)$ for $0 \leqslant T \leqslant 2\pi\sqrt{m_2}$, we note that only in the left end of this interval, that is, only for $0 \leqslant T \leqslant 2\sqrt{2}\sqrt{\frac{m_2}{m_1}}$, does formula (25) give 'certainly appropriate' behavior of the function $U_1(T)$: in this end of the interval the 'bracket' from (25) takes $\approx \frac{2\sqrt{2}}{\pi}\frac{m_2}{\sqrt{m_1}}$ values; each of them is approximately equal to $\pm\sqrt{2\pi m_1}$. When T increases from $2\sqrt{2}\sqrt{\frac{m_2}{m_1}}$ to $\sqrt{8\pi^2\frac{m_2}{m_1}\ln m_2}$, the values $|U_1(T)|$ decrease sharply, and for the rest of the values of T up to $2\pi\sqrt{m_2}$ the asymptotics (25) will give only the estimate of $|U_1(T)|$ by a small value.

3.4. *Characteristics of a packet of oscillations*

We now consider all intervals $0 \leqslant T \leqslant m_2^{\frac{5}{6}}$ under investigation. We cover it by intervals of the form

$$k4\pi\sqrt{m_2} - 2\pi\sqrt{m_2} \leqslant T \leqslant k4\pi\sqrt{m_2} + 2\pi\sqrt{m_2}, \qquad k = 1, 2, \ldots, k_1 = \left[\frac{1}{4\pi}m_2^{1/3}\right].$$

Let $T = k4\pi\sqrt{m_2} + x$, $|x| \leqslant 2\pi\sqrt{m_2}$. Then

$$\gamma = \left\|\frac{T}{4\pi\sqrt{m_2}}\right\| = \frac{|x|}{4\pi\sqrt{m_2}}. \tag{32}$$

We know that for the asymptotics (25) to be rich in content it is necessary that inequality (26) is valid or, taking into account (32), the inequality

$$\frac{|x|}{4\pi\sqrt{m_2}} \leqslant \sqrt{4m_1\ln m_2\left(\frac{1}{4\pi^2 m_1^2} + \frac{T^2}{64\pi^2 m_2^3}\right)} \tag{33}$$

holds. Increasing the right-hand side of inequality (33) by a little, namely substituting T by a slightly greater value, not depending on x,

$$T \leqslant k4\pi\sqrt{m_2} + 2\pi\sqrt{m_2} = 4\pi\sqrt{m_2}\left(k + \frac{1}{2}\right),$$

we obtain that only for $|x| \leqslant \Upsilon_1(k)$, where

$$\Upsilon_1(k) = \sqrt{64\pi^2 m_1 m_2 \ln m_2\left(\frac{1}{4\pi^2 m_1^2} + \frac{\left(k+\frac{1}{2}\right)^2}{4m_2^2}\right)}, \tag{34}$$

can formula (25) be meaningful. It is easily seen from (34) that if m_1 is 'great enough' ($m_2 > m_1$) and a is less than $\approx \exp(m_1)$, then

$$2\pi\sqrt{m_2} > \Upsilon_1(k).$$

On the other hand, it is easy to check that for $|x| \leqslant \Upsilon_0(k)$, where

$$\Upsilon_0(k) = \sqrt{64\pi^2 m_1 m_2\left(\frac{1}{4\pi^2 m_1^2} + \frac{\left(k-\frac{1}{2}\right)^2}{4m_2^2}\right)},$$

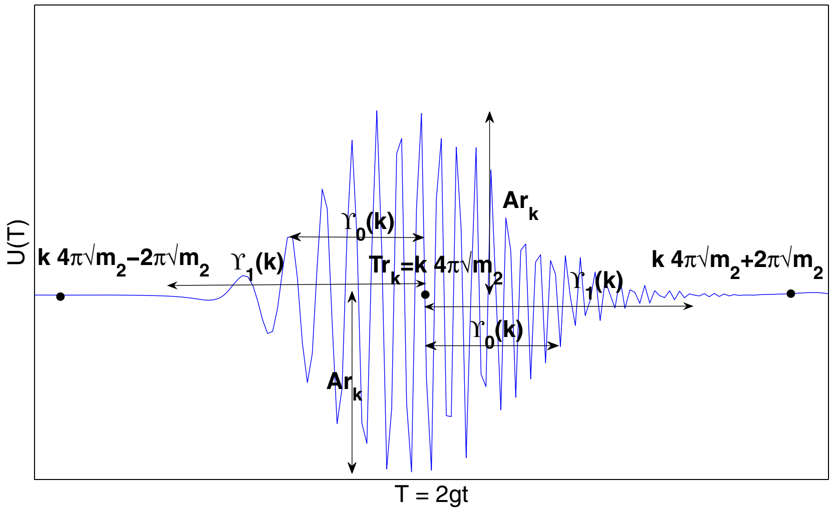

Figure 1. Scheme of the kth packet of oscillations: $m_2 = m_1 + a + 1, a = (\Delta/(2g))^2$, $m_1 = |\alpha|^2$, $Tr_k = k4\pi\sqrt{m_2}$; $Ar_k \approx \left(\frac{1}{4\pi^2 m_1^2} + \frac{k^2}{4m_2^2}\right)^{-\frac{1}{4}}$, $\Upsilon_0(k) = \left(64\pi^2 m_1 m_2\left(\frac{1}{4\pi^2 m_1^2} + \frac{(k-\frac{1}{2})^2}{4m_2^2}\right)\right)^{\frac{1}{2}}$, $\Upsilon_1(k) = \sqrt{64\pi^2 m_1 m_2 \ln m_2\left(\frac{1}{4\pi^2 m_1^2} + \frac{(k+\frac{1}{2})^2}{4m_2^2}\right)}$.

the inequality $D\gamma^2 \leqslant 1$ holds, that is, $\exp(-1) \leqslant \exp(-D\gamma^2) \leqslant 1$. Moreover, for $T = 4\pi k\sqrt{m_2} + x$, $|x| \leqslant \Upsilon_0(k)$ the factor $\left(\frac{1}{4\pi^2 m_1^2} + \frac{T^2}{64\pi^2 m_2^3}\right)^{-\frac{1}{4}}$ differs a little from the number $\left(\frac{1}{4\pi^2 m_1^2} + \frac{k^2}{4m_2^2}\right)^{-\frac{1}{4}}$. It remains to understand the behavior of the argument of the cosine as a function of x, $|x| \leqslant \Upsilon_0(k)$. To do so, we consider the following function:

$$f(x) = \Phi(k4\pi\sqrt{m_2} + x) = x\sqrt{m_2} + (k4\pi\sqrt{m_2})\sqrt{m_2} - \frac{1}{2}\arctan\frac{(k4\pi\sqrt{m_2} + x)m_1}{4m_2^{\frac{3}{2}}} - \frac{(k4\pi\sqrt{m_2} + x)^3}{128\pi^2 m_2^{\frac{5}{2}}}\left(\frac{1}{4\pi^2 m_1^2} + \frac{(k4\pi\sqrt{m_2} + x)^2}{64\pi^2 m_2^3}\right)^{-1}.$$

According to the Lagrange formula, we find

$$f(x+h) - f(x) = h\sqrt{m_2} + hr_1, \qquad |r_1| \leqslant \frac{m_1}{6m_2^{\frac{3}{2}}}.$$

Consequently on the interval $-\Upsilon_0(k) \leqslant x \leqslant \Upsilon_0(k)$ the cosine will take values +1 and $-1 \approx \frac{2}{\pi}\Upsilon_0(k)\sqrt{m_2}$ times: $\Upsilon_0(k)$ defines the width of the time interval of 'notable' oscillations of the kth collapse/revival. For $\Upsilon_0(k) \leqslant |x| \leqslant \Upsilon_1(k)$, the principal term decreases when x increases from the factor $\exp(-D\gamma^2)$ (the cosine oscillations are continued): $\Upsilon_1(k)$ defines the width of the time interval of the kth collapse/revival oscillation ($2\Upsilon_1(k)$ is the general width of the packet). For $\Upsilon_1(k) \leqslant x \leqslant 2\pi\sqrt{m_2}$ the principal term becomes a small one, and instead of the asymptotics we get only the estimate of $|U_1(T)|$, which is given by the remainder term.

An approximate scheme of the kth packet is presented in the plot in figure 1 (all the plots of this paper are built with use of program Matlab 7).

Note that the maximal value of the kth packet oscillations, which we will call the 'amplitude' of the kth revival, is equal to

$$Ar_k \approx \left(\frac{1}{4\pi^2 m_1^2} + \frac{k^2}{4m_2^2}\right)^{-\frac{1}{4}},$$

that is, it decreases when k increases.

4. Conclusion

4.1. Main asymptotic formula for the atomic inversion

The value A, defined by (3), is easily calculated from (5) and (8), (10), (15):

$$A = \frac{a}{m_1 + a + 1} + \frac{14\theta}{\sqrt{2\pi m_1}}\left(1 + \frac{a}{m_1}\right)^{-1}.$$

Thus, we finally get the following asymptotic formula for the atomic inversion:

for $0 \leqslant T \leqslant (m_1 + a + 1)^{\frac{5}{6}}$,

$$\begin{aligned} W(t) = U(T) = {} & \frac{a}{m_1 + a + 1} + \frac{m_1 + 1}{m_1 + a + 1}\left(1 + \frac{T^2 m_1^2}{16(m_1 + a + 1)^3}\right)^{-\frac{1}{4}} \\ & \times \exp\left(-2\pi^2 m_1\left(1 + \frac{T^2 m_1^2}{16(m_1 + a + 1)^3}\right)^{-1}\left(\left\|\frac{T}{4\pi\sqrt{m_1 + a + 1}}\right\|\right)^2\right) \\ & \times \cos\left(\frac{\pi^2}{2} T \frac{m_1^2}{\sqrt{(m_1 + a + 1)^3}}\left(1 + \frac{T^2 m_1^2}{16(m_1 + a + 1)^3}\right)^{-1}\left(\left\|\frac{T}{4\pi\sqrt{m_1 + a + 1}}\right\|\right)^2\right. \\ & \left. - \frac{1}{2}\arctan\frac{T m_1}{4\sqrt{m_1 + a + 1}} - T\sqrt{m_1 + a + 1}\right) \\ & + \theta\left(\frac{2}{\sqrt{2\pi m_1}}\left(1 + \frac{a}{m_1}\right)^{-1}\left(T m_1^{\frac{3}{2}}(m_1 + a + 1)^{-\frac{5}{2}} + 19\right)\right), \end{aligned} \tag{35}$$

where $a = \frac{\Delta^2}{4g^2}$, $T = 2gt$, $m_1 = |\alpha|^2$. Above we showed that on the interval $0 \leqslant T \leqslant (m_1 + a + 1)^{\frac{5}{6}}$ approximately $\frac{1}{4\pi}(m_1 + a + 1)^{\frac{1}{3}} + 1$ packets of quantum oscillations appear, which are localized on the intervals of the form

$$4\pi\sqrt{m_1 + a + 1}\left(k - \tfrac{1}{2}\right) \leqslant T \leqslant 4\pi\sqrt{m_1 + a + 1}\left(k + \tfrac{1}{2}\right),$$

where $k = 0, 1, 2, \ldots, k_1$ and $k_1 = \left[\frac{(m_1+a+1)^{\frac{1}{3}}}{4\pi}\right]$.

Moreover, the behavior of the packet of oscillations has a distinctly asymptotic character on the intervals of the form

$$\begin{aligned} 4\pi\left(k\sqrt{m_2} - \sqrt{\left(\frac{m_2}{m_1} + \frac{m_1}{m_2}\frac{\left(k + \frac{1}{2}\right)^2}{\pi^2}\right)\ln m_2}\right) & \leqslant T \\ & \leqslant 4\pi\left(k\sqrt{m_2} + \sqrt{\left(\frac{m_2}{m_1} + \frac{m_1}{m_2}\frac{\left(k + \frac{1}{2}\right)^2}{\pi^2}\right)\ln m_2}\right), \end{aligned}$$

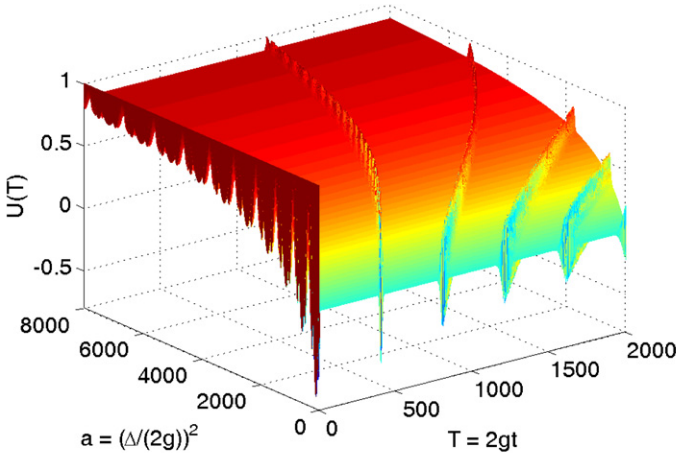

Figure 2. The principal term of the approximation of the inversion $U(T)$ for the average photon number $m_1 = |\alpha|^2 = 1024$; the time position of peaks is defined by $Tr_k = k4\pi\sqrt{a + 1025}$.

where $m_2 = m_1 + a + 1$, $k = 0, 1, 2, \ldots, k_1$ and $k_1 = \left[\frac{m_2^{\frac{1}{3}}}{4\pi}\right]$, and the values that the packet oscillations take are the highest by the absolute value on the subintervals of the form

$$4\pi\left(k\sqrt{m_2} - \sqrt{\left(\frac{m_2}{m_1} + \frac{m_1}{m_2}\frac{\left(k-\frac{1}{2}\right)^2}{\pi^2}\right)}\right) \leqslant T$$
$$\leqslant 4\pi\left(k\sqrt{m_2} + \sqrt{\left(\frac{m_2}{m_1} + \frac{m_1}{m_2}\frac{\left(k-\frac{1}{2}\right)^2}{\pi^2}\right)}\right),$$

where $m_2 = m_1 + a + 1$, $k = 0, 1, 2, \ldots, k_1$ and $k_1 = \left[\frac{m_2^{\frac{1}{3}}}{4\pi}\right]$.

4.2. The limits of the availability of the main asymptotics

Note that formula (35) can be an effective one and reflects correctly the essence of event also for much smaller values of m_1 than we assumed above according to condition (6), since all estimates are much stronger than what we used above.

At the same time, since approximately $\frac{1}{4\pi}(m_1 + a + 1)^{\frac{1}{3}}$ revivals appear in 'the proved' region (7), then in order to observe at least the first revival in this region, the sum of parameters $m_1 + a$ must be more than $\approx 64\pi^3 - 1$. This confirms the plots presented in figures 2 and 3: the first revival appears when $a \gtrapprox 960$ and when $m_1 \gtrapprox 1759$ respectively (the plot in figure 3 is built for m_1 beginning from $m_1 = 1600$).

We see that for small values of $m_1 + a$, we have proved the asymptotical formula for the first packet (first collapse) only.

As was already mentioned in subsection 1.3 by making more precise bounds, it is possible to extend the time limits of the availability of the proved formulae defined by (7), say up to $0 \leqslant T \leqslant C_0(m_1 + a + 1)$, where C_0 is a constant. Then by verifying separately the groups of certain values a and m_1, it is possible to make more precise bounds for C_0 (for small parameters a and m_1, C_0 can be relatively great). The reason to assume such a possibility for small values $a + m_1$ is due to the numerical calculations presented in three pairs of plots in

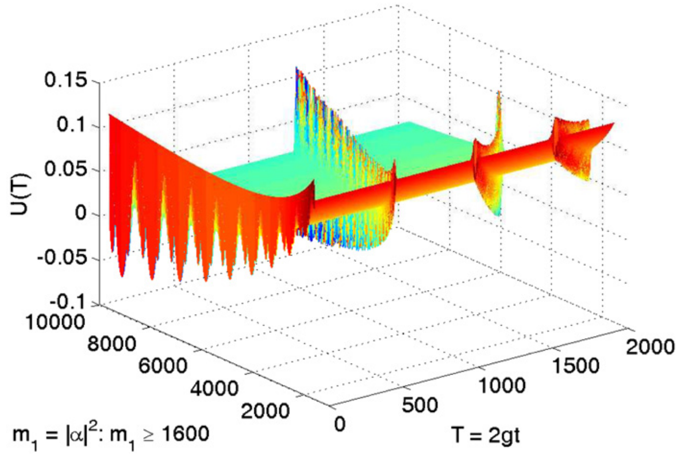

Figure 3. The principal term of the approximation of the inversion $U(T)$ for the quadratic 'scaled detuning parameter' $a = (\Delta/(2g))^2 = 225$; the time position of peaks is defined by $Tr_k = k4\pi\sqrt{m_1 + 226}$.

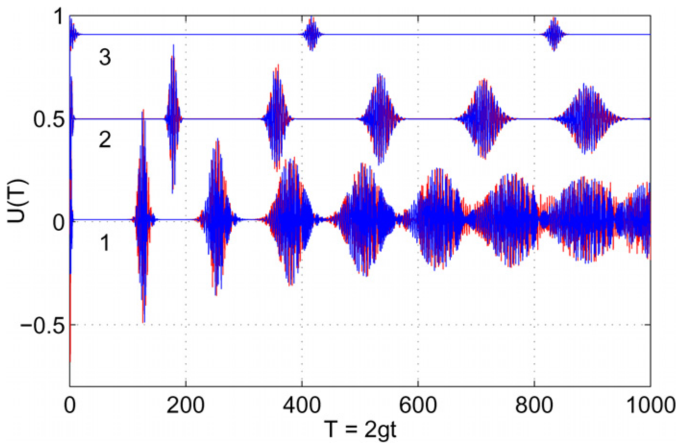

Figure 4. Sum of the first 155 terms of the series (4) (red line) and the principal term of the approximation (35) (blue line) of the inversion $U(T)$ for the average photon number $m_1 = |\alpha|^2 = 100$ and the quadratic 'scaled detuning parameter' $a = (\Delta/(2g))^2$ with 1: $a = 1$, 2: $a = 100$, 3: $a = 1000$.

figure 4. It is easily seen that certain characteristics of the packets of quantum oscillations: the positions of the packets, the positions of the peaks of the packets, maximal by absolute value oscillation of a packet, coincide on the time intervals which are much longer than proved by (7).

However, in this paper, we prove rigorously the accuracy of the approximation only on the interval $0 \leqslant T \leqslant (m_1 + a + 1)^{\frac{5}{6}}$. In a general case, if $T > m_1 + a + 1$, then for a rigorous proof one needs to apply the theorem on the approximation of a trigonometric sum by a shorter one—ATS (see [7, 9–14]; see especially [8]). In this way, the function $W(t)$ will be already approximated by a sum of a few cosines, the number of which will increase with increasing length of the interval of variation of *T*.

4.3. The important role of the quadratic 'scaled detuning parameter'

Since in all the above cases our estimates were uniform in m_1 and $m_2 = m_1 + a + 1$, from the formulae obtained one can see how the behavior of the atomic inversion depends on the value of a, $a = \frac{\Delta^2}{4g^2}$.

If a is of order less than or equal to m_1, that is, $0 \leqslant a \leqslant C_1 m_1$, where $C_1 \geqslant 0$ is a constant, then $W(t)$ behaves (on the whole) in the same way as it would for $a = 0$, that is, for $\Delta = 0$. If a is greater than m_1 by order, for example $a = m_1 \ln m_1$ or $a = m_1^{\frac{3}{2}}$, then the behavior of $W(t)$ will be essentially different from the resonance case; the number of local extrema changes, the extremum values will be different and so on.

As can be seen in the plots in figure 4, with increasing a the axis of symmetry of the graph tends to line $y = U(T) = 1$, and the 'amplitude of revivals' decreases. When a becomes greater than m_1, all values of the inversion become positive (see also figure 2). Such a change in the inversion behavior corresponds to the phenomenon of the 'dispersive limit' (see [2, 18, 19]), as a result of which the interaction of an atom with the nonresonant field introduces a dispersive shift in the atomic states, which is not accompanied by transitions in the system, that is, the number of photons in the system does not change in this case.

4.4. Concluding remarks. Quantum effects and number theory

Asymptotic formulae proved above reflect the peculiarities of the behavior of the inversion observed in experiments: collapses and revivals of quantum oscillations are repeated, the amplitude of the oscillations decreases with time and the duration of revivals increases. Besides, the moments when the collapses and revivals take place, and their amounts on the intervals under consideration, are determined by the relation between the parameters of the system: the detuning parameter, the atom–field coupling constant and the initial average number of photons before the interaction of the field with the atom.

The revivals in the JCM reflect the discrete structure of the photon distribution in this model, which is a pure quantum mechanical phenomenon. The discreteness, expressed in the Jaynes–Cummings sum for the atomic inversion, was the basis to apply to the approximation of this sum the techniques which were developed in number theory, which is the field of mathematics whose main objects of investigation are discrete objects (integers).

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